Reglog – the game

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MIT ACT seminar
## Outline

1. **Introduction**
   - Playing with logic
   - The chase
   - Plan for the talk

2. The math

3. Reglog – the games

4. Conclusion
Introduction

Playing with logic

Minority Report
The following has very little direct relation to the movie Minority Report; it’s just an analogy for color.
The 2002 movie *Minority report* showed detective Tom Cruise playing seamlessly with logic.

- A computer database held relevant information.
- Cruise could pull it up and manipulate it to solve crimes.
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Let’s imagine our own version of a detective scenario.

*Brought to you by... regular logic – the game!*
Working with logic

- Imagine you are an investigator on a case.
- You’re adding to and narrowing down your set of suspects.
- You can pull up cells from the computer’s database.
Imagine you are an investigator on a case.
You’re adding to and narrowing down your set of suspects.
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and define POI (person of interest) as the result:
Adding beliefs

You can add beliefs about the world.

Belief: “If two persons work at the same startup, they are acquainted.”
Adding beliefs

You can add beliefs about the world.

Belief: “If two persons work at the same startup, they are acquainted.”

Belief: “In any case, my suspect is acquainted with the victim.”
Accessing data

You can click a cell to see what’s inside:

Persons are internal identifiers; we want to see facts about the victims.
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<table>
<thead>
<tr>
<th>victim fact</th>
</tr>
</thead>
<tbody>
<tr>
<td>person</td>
</tr>
<tr>
<td>P105</td>
</tr>
<tr>
<td>P820</td>
</tr>
</tbody>
</table>
Accessing data

You can click a cell to see what’s inside:

Persons are internal identifiers; we want to see facts about the victims.

Some knowledge is missing or otherwise imperfect.
Reasoning

The machine knows basic logical reasoning.

Manipulate diagrams by ...

- ... combining or breaking up intersectionalities as above;
Reasoning

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Manipulate diagrams by ...

- ... combining or breaking up intersectionalities as above;
- ... breaking dots:

\[
\{(x, y, z) \mid x = y = z\} \subseteq \{(x, y, z) \mid x = y\}
\]
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Reasoning: regular “old” logic, with a shiny new math-specified GUI.
The chase is on

You’ve identified certain sources of and constraints on your suspect

- Sue is a suspect.
- The suspect is acquainted with the victim.
- (and so on.)
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To catch your suspect, you must use the venerable chase...

\[ \forall (p, p'): P \exists (c: C). W(p, c) \land W(p', c) \land S(c) \vdash \exists (t: T). A(t, p, p') \]

also known as: regular logic sequents, embedded dependencies, existential horn clauses, lifting problems.

The chase minimally "repairs" \( I \rightarrow I' \), with \( I' \) conforming to axioms.
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To complete the detective story:

- You have created an important cell: locations the suspect may be in.
- You hook it up to your car’s autonomous driver, and off you go.
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The End.
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The End.

Can we make this real?
“Detective” is not the only game in reglog – the game.

- I’ll briefly discuss the mathematics involved.
- I’ll talk about how it connects to the GUI described above.
- I’ll end by giving several other games, besides “detective”.

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1 Introduction

2 The math
- Regular categories
- Graphical regular logic
- Mathematical spec of the GUI
- Connection with CQL

3 Reglog – the games

4 Conclusion
Regular logic and regular categories

Regular logic is the internal logic of regular categories.

- Regular categories are categories $\mathcal{R}$ with
  - finite limits (terminal object 1 and pullbacks), and
  - pullback-stable image factorizations.
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- Regular categories are categories $\mathcal{R}$ with
  - finite limits (terminal object 1 and pullbacks), and
  - pullback-stable image factorizations.
- Say $\mathcal{R}$ is fully-inhabited if $\mathcal{R}(1, r) \neq \emptyset$ for each $r \in \mathcal{R}$.
  - Set is a regular category, but not fully inhabited.
  - The category of pointed sets is fully inhabited.
  - Have categories $\text{RegCat}_* \subseteq \text{RegCat}$ of (fully-inhabited) regular cats.

Examples of regular categories:

- Set, and more generally any topos;
- $\text{Set}^{\text{op}}$, opposite of any topos, also $\text{TopSp}^{\text{op}}$;
- The category of models of any Lawvere theory (Groups, Rings, ...);
- The slice (also the coslice) of any regular category over any object;
- Exponential ideal: if $\mathcal{R}$ regular and $\mathcal{C}$ a category, then $\mathcal{R}^\mathcal{C}$ is regular.
How to think of regular categories

Regular categories $\mathcal{R}$ are those with a good *bicategory of relations*.

- A relation in $\mathcal{R}$ is a subobject $S \subseteq A \times B$.
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Regular categories have enough structure to do regular logic.
Graphical regular logic

Think of regular categories as having good notion of relation.
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- \textbf{Regular logic} is the logic of relations in regular categories.
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- **Regular logic** is the logic of relations in regular categories.
- Given rel’ns $R(x, y)$ and $S(y, z)$, can interpret: $\exists y. R(x, y) \land S(y, z)$. 
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- Given rel’ns $R(x, y)$ and $S(y, z)$, can interpret: $\exists y. R(x, y) \land S(y, z)$.
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  - Have a cell (filled $\Gamma$-shaped shell) for every subobject $c \subseteq r_1 \times \cdots \times r_n$.
  - Wiring diagrams denote combinations of finite limits and images.
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- Let’s discuss wiring diagrams.
Mathematically, what is a wiring diagram \((\Gamma_a, \Gamma_b, \Gamma_c) \rightarrow \Gamma\).
Mathematical specification of wiring diagrams

Mathematically, what is a wiring diagram \((\Gamma_a, \Gamma_b, \Gamma_c) \rightarrow \Gamma\) ?

It is a morphism \((\Gamma_a, \Gamma_b, \Gamma_c) \rightarrow \Gamma\) in the operad of corelations.

- Another viewpoint: it is an equivalence relation on \(\Gamma_a \sqcup \Gamma_b \sqcup \Gamma_c \sqcup \Gamma\).
Wiring diagrams as logical expressions

We can convert a wiring diagram like this into a logical expression:

\[
Q(u_3, v_4, w, x) \land R(w, y_5) \land S(x, y_5, z_6) \land (t_1 = t_2)
\]
Wiring diagrams as logical expressions

We can convert a wiring diagram like this into a logical expression:

- Write type of exterior shell, naming each port by a distinct variable.
- Write quantifier $\exists (x : X)$ for each unexposed wire of type $X$.
- AND together internal cells, with established var. names from above.
- Equate variables for exposed ports that are connected.

$$O(t_1,t_2,u_3,v_4,y_5,z_6):=\exists (w : W,x : X). Q(u_3,v_4,w,x) \land R(w,y_5) \land S(x,y_5,z_6) \land (t_1=t_2)$$
Our “minority report” detective GUI can be understood as follows.

- Fix a set $T$ (each $t \in T$ is a string label: person, height, etc.).
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  - Objects: arities $n \overset{t}{\to} T$, i.e. lists $\Gamma = (t(1), \ldots, t(n)) \in T^n$.
  - Monoidal structure: concatenate lists.
  - 1-morphisms $(n_1, t_1) \to (n_2, t_2)$: partitions of $n_1 + n_2$, respecting types
  - 2-morphisms: refinement (discrete partition is largest).
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  - Monoidal structure: $(1, \times)$.
- We will consider certain functors $\Phi : \mathcal{Corel}_T \rightarrow \mathcal{Poset}$.
  - To each shell $\Gamma \in \mathcal{Corel}_T$, a poset $\Phi(\Gamma)$.
  - We denote the order in $\Phi(\Gamma)$ using the logical entailment symbol $\Rightarrow$. 
Formal specification of graphical calculi II

We have monoidal 2-categories \( \mathbb{Corel} \) and \( \mathbb{Poset} \).

**Definition**

An *(inhabited)* regular calculus is a lax monoidal 2-functor 

\[
\Phi : \mathbb{Corel}_T \to \mathbb{Poset}
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such that the laxators are right adjoints.
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Our terminology: *ajax* monoidal functors: the laxators

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Regular calculi and regular categories

Denote by $\mathcal{Corel}$-Alg the category of inhabited regular calculi

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**Theorem**

There is an adjunction

$$\text{Corel-Alg} \xleftrightarrow[\Phi]{\Psi} \text{RegCat}_\ast,$$

such that for any fully-inhabited regular category $\mathcal{R}$, the counit $\Psi(\Phi(\mathcal{R})) \to \mathcal{R}$ is an equivalence of categories.
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A similar theorem holds when $\text{RegCat}_*$ is replaced by $\text{RegCat}$: arxiv.org/abs/1812.05765.
Aside: operadic vs. monoidal

There are two versions of the whole story.

- Operadic version: nice pictures.
- Monoidal version: more familiar, better notation.
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The translation uses “shells in a trench coat”.
Mathematical spec of the GUI

We want software to do the detective work; call it “Reglog – the game.”
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  - Morphisms: drawn as wiring diagrams, encoded as partitions.
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  - To each shell $\Gamma$, supply elts of $\Phi(\Gamma)$, drawn as cells (fillers).
  - Supply definition of $\varphi_1 \leq \varphi_2$, drawn perhaps as $\varphi_1 \vdash \varphi_2$.
  - To each wiring diagram $w$, supply monotonic function $\Phi(w): \Phi(\Gamma_1) \times \cdots \times \Phi(\Gamma_n) \to \Phi(\Gamma')$.
  - Ensure $\Phi$ preserves composition, identity, and dot-breaking.
The math

Connection with CQL

Backend: open source CQL

Backend: categorical query language, another approach to databases.

A schema $S$ is roughly a category and looks like this:

Employee
• Department
• String
◦ $\ast$
◦ $\ast$

WorksIn

FName
Mngr
Secr
DName
Bdgt

An instance is roughly a functor $I: S \to \text{Set}$.

Given schema $S$, consider each dot $s \in S$ as a shell and as a type.

Wires of shell $s$'s outgoing arrows $s \to t$ typed by target $t$.

If $I$ is an $S$-instance, get filling cells as in detective case.

Each regular sequent $\phi(\vec{x}) \vdash \psi(\vec{x})$ is called an embedded dependency.

Chase these EDs to "repair data", forcing the axioms to hold.
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![Diagram of schema S]

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![Diagram of Employee, Department, and their relationships]

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3. Reglog – the games
   - Common interface, many games
   - Representing and interacting with knowledge
   - Logic games
   - Systems

4. Conclusion
“Reglog – the game” is actually a bunch of games.

- In common: same GUI and common forms of interaction
  - Create new shells, attach shells, compose wiring diagrams
  - Zoom in and out of wiring diagrams.
  - Fill shells with cells
  - Click on cells (filled in shells) for interaction.
  - Do reasoning (entailment, substitution, dot-breaking)
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- Differences between different games $\Phi$:
  - Each $\Phi$ gives a world of cells that inhabit the shells.
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- Differences between different games $\Phi$:
  - Each $\Phi$ gives a world of cells that inhabit the shells.
- Some examples:
  - Where are my keys?
  - Smart witter
  - Partitions game
  - Boolean circuits
  - Solve equations
  - etc.
Game: Where are my keys?

A simpler version of detective game: no reasoning $\vdash$, just query.

- Tell Alexa or Siri facts as you get them, and sort your life.
Game: Where are my keys?

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<tr>
<td>keys</td>
<td>top drawer</td>
<td>2019/04/08</td>
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- Query using the graphical interface
  - Attach thing keys yesterday date to find places your keys were yesterday.
  - Similar interface for your calendar, your address book, etc.
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  - Cardiologists who fit my availability and who take my insurance.
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- Use public information repos.
  - Cardiologists who fit my availability and who take my insurance.
  - There is an internet-published “doctor” cell.
  - Connect \(\text{specialty} \rightarrow \text{cardiologist}\) to doctor’s speciality port.
  - Connect its availability and insurance ports to your own.
  - Output doctor’s name and phone number.
Game: Smart witter

A smarter social network.

- This is another simplification of the detective game.
- Difference: cells are just named, not inhabited.
- Idea: people blog concepts built from simple pieces.
Game: Smart witter

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  - Others can disassemble, reassemble, and nick-name concepts
  - Zoom into others’ concepts to see what they really mean.


The machine finds concepts that people keep reusing.

Logic gates are particular wirings of transistors.

Adder circuits are particular wirings of logic gates.

Out of all configurations, some are very popular, i.e. reused.

Same idea for human concepts; find reusable ideas (memes).
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  - Same idea for human concepts; find reusable ideas (memes).
Game: Partitions puzzle

Put pieces: into puzzle: to obtain:

1
2
3
I’ll give you a minute to solve it.
Game: partitions puzzle (solution)
Game: Boolean circuits

Boolean circuits are special cases of boolean relations.

- What are boolean relations?
  - A boolean relation is a subset of $B^n = \{\text{true, false}\}^n$ for some $n$.
  - Familiar: AND, OR, IMPLIES, NOT, TRUE, FALSE.
**Game: Boolean circuits**

Boolean circuits are special cases of boolean relations.

- **What are boolean relations?**
  - A boolean relation is a subset of $\mathbb{B}^n = \{\text{true}, \text{false}\}^n$ for some $n$.
  - Familiar: AND, OR, IMPLIES, NOT, TRUE, FALSE.
  - Those are basic circuits; they’re functions.
  - Consider also relations, like $\leq$.

<table>
<thead>
<tr>
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<tr>
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Puzzles: build up complex relation $S$ using simple parts $R_1,...,R_n$. Example: $\leq$, IMPLIES, TRUE.
Game: Boolean circuits

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- Puzzles: build up complex relation $S$ using simple parts $R_1, \ldots, R_n$.
- Example: $\leq = \text{IMPLIES} \quad \text{TRUE}$
Game: Boolean circuits

Boolean circuits are special cases of boolean relations.

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- Puzzles: build up complex relation $S$ using simple parts $R_1, \ldots, R_n$.

- Example: $\leq = \text{IMPLIES} - \text{TRUE}$
Game: Solving equations

Consider an arbitrary system of equations having the following form:

\[
\begin{align*}
    f_1(t, u, v) &= 0 \\
    f_2(v, w, x) &= 0 \\
    f_3(u, w, x, y) &= 0 \\
    f_4(x, z) &= 0
\end{align*}
\]

Bold variables are those we want to expose; others are latent or unexposed.
Game: Solving equations

Consider an arbitrary system of equations having the following form:

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\[ f_3(u, w, x, y) = 0 \]
\[ f_4(x, z) = 0 \]

Bold variables are those we want to expose; others are latent or unexposed.

Said another way, we want \( \{(t, v, z) \mid \exists u, w, x, y : f_1 = f_2 = f_3 = f_4 = 0\} \).
Systems of equations via pixel arrays

Consider just two equations \( f(x, w) = 0 \) and \( g(x, y) = 0 \).

- Plot each in its own bounding box, say in the range \([-1.5, 1.5]\).
- Consider the plots as matrices \( M, N \) whose entries are on/off pixels.
- That is, \( M \) and \( N \) are boolean matrices corresponding to \( f \) and \( g \).

Multiplying these two matrices \( MN \) yields the simultaneous solution.
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Multiplying these two matrices \( MN \) yields the simultaneous solution.
- For example, plot equations \( x^2 = w \) and \( w = 1 - y^2 \), and multiply.

\[
\begin{align*}
\text{a.} & & \text{N: pixelMatrix( w = 1-y^2)} \\
\text{b.} & & \text{ordinary matrix product, M*N} \\
\text{c.} & & \end{align*}
\]
A more complex example

The following eq's are not differentiable, nor even defined everywhere.

\[
\begin{align*}
\cos \left( \ln(z^2 + 10^{-3}x) \right) - x + 10^{-5}z^{-1} &= 0 \\
cosh(w + 10^{-3}y) + y + 10^{-4}w &= 2 \\
\tan(x + y)(x - 2)^{-1}(x + 3)^{-1}y^{-2} &= 1
\end{align*}
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(Equation 1)  (Equation 2)  (Equation 3)

Q: For what values of \( w \) and \( z \) does a simultaneous solution exist?
A more complex example

The following eq's are not differentiable, nor even defined everywhere.

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\]
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\tan(x + y)(x - 2)^{-1}(x + 3)^{-1}y^{-2} = 1 \quad \text{(Equation 3)}
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Q: For what values of \(w\) and \(z\) does a simultaneous solution exist?
**Game: systems of linear equations**

PA is the backend for a game that plots sol’ns to arbitrary systems.
Game: systems of linear equations

PA is the backend for a game that plots sol’ns to arbitrary systems.

- Similar game: each cell $f_i$ is a linear eq’n, e.g. $x_1 + 3x_2 - 2x_4 = 0$.
- Then the outer cell is a linear equation too.
- “Exponentially” smaller matrices to multiply.
Game: theory of a group, ring, etc.

Monoids, groups, rings, modules: each has an associated algebraic theory.

- Choose any algebraic theory, or more generally regular theory.
Game: theory of a group, ring, etc.

Monoids, groups, rings, modules: each has an associated algebraic theory.

- Choose any algebraic theory, or more generally regular theory.
  - Let’s say the theory of monoids.
    - Have a multiplication cell $\ast$ and a unit cell $e$.
    - These satisfy various equations (more generally, regular axioms).
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    - Game designer could program in things like $x \ast e = x$ simplifications.
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- Within the game, create new cells and manipulate them.
  - E.g. add orthogonal and transpose and axiomatize them.

\[
\text{transpose} \ast e = \text{orthogonal}
\]
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- Within the game, create new cells and manipulate them.
  - E.g. add \( \text{orthogonal} \) and \( \text{transpose} \) and axiomatize them.

Like “fold-it” for protein folding, players can help w/o understanding.
Simulink (makers of Matlab): model and simulate dynamic systems.

Connect up smaller dynamic systems.
Game: Simulink

- Simulink (makers of Matlab): model and simulate dynamic systems.

- Connect up smaller dynamic systems.
- Each can be understood as a relation in the temporal topos (TTT).
- Use reglog – the game as an interface for Simulink.
Let’s make this real!

We’re ready to make this happen.

- The background math is complete
  - We understand the data structures involved.
  - Experience shows that coding it will uncover hidden assumptions.
  - Example: maybe player can “transform” typeset $T$ and $\Phi$, mid-game.
- We’ve seen some example games and roughly how they’d work.
- Creativity required to precise game specs and to create new games.
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How to proceed?

- I have some funding to get a programming effort started.
- Please contact me or Brendan if you’re interested in contributing.
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*Thanks!*