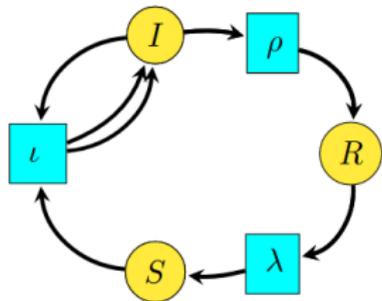


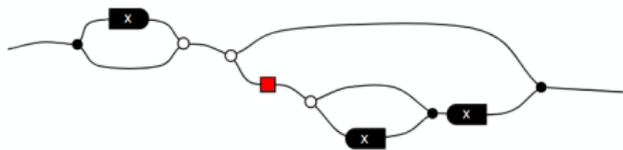
# Hypergraph categories as cospan algebras

Brendan Fong, with David Spivak

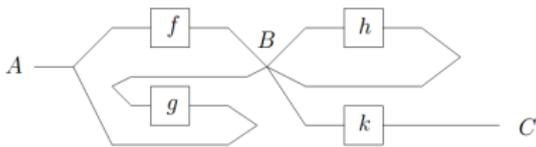
Category Theory 2018  
University of Azores  
10 July 2018



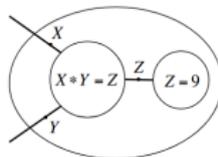
Baez, Pollard: *A compositional framework for reaction networks*



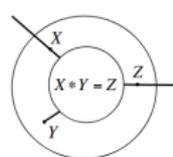
Bonchi, Sobocinski, Zanasi: *A categorical semantics of signal flow graphs*



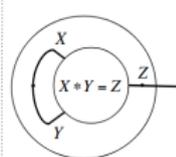
Rosebrugh, Sabadini, Walters: *Calculating colimits compositionally*



"all pairs of integers whose product is 9"



"all pairs of integers in which one is divisible by the other."



"all perfect squares"

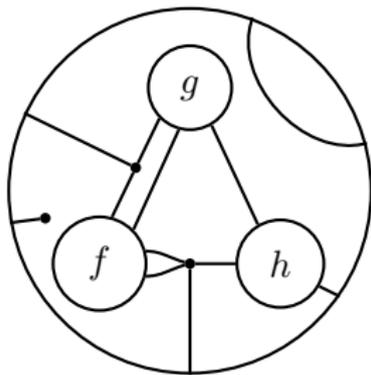
Spivak: *The operad of wiring diagrams*

# Outline

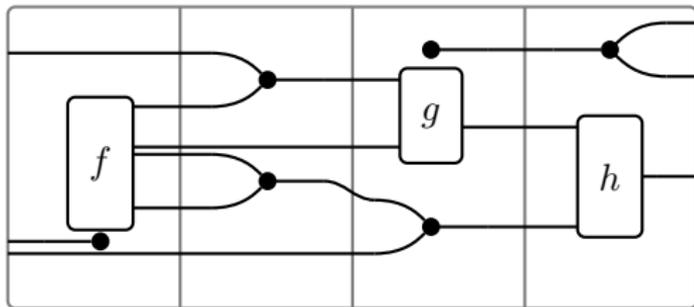
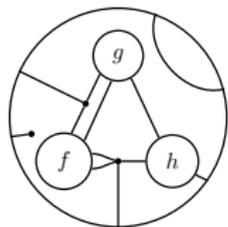
- I. Hypergraph categories
- II. Cospan algebras
- III. The equivalence

# I. Hypergraph categories

Abstractly, how do we construct this?

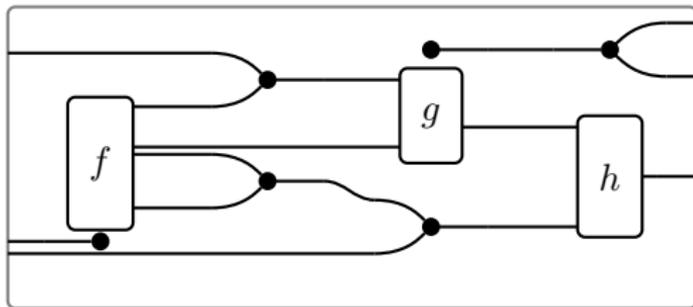
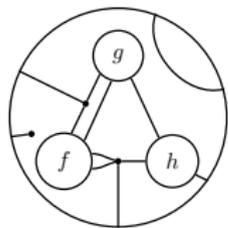


... as structured monoidal category

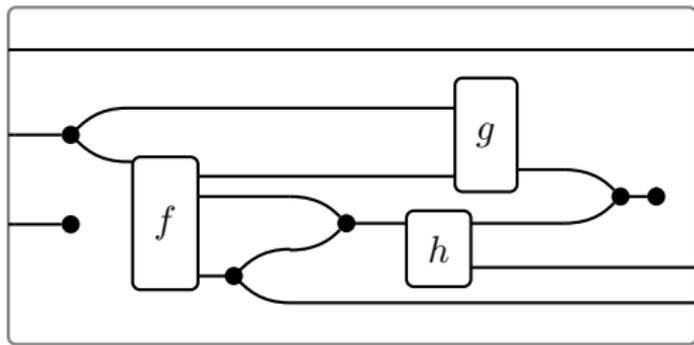
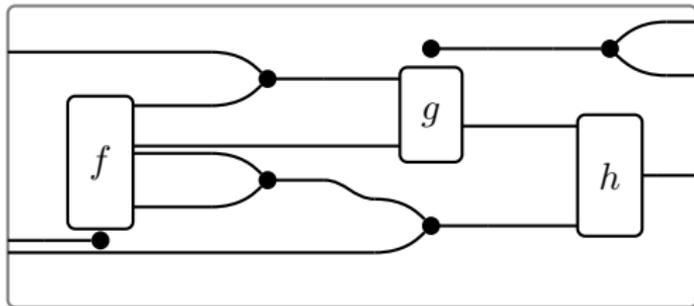
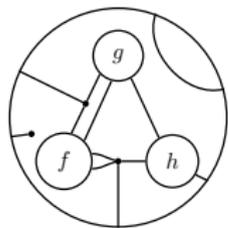


$$(1 \otimes f \otimes \rightarrow \otimes 1); (\rightarrow \otimes 1 \otimes \rightarrow \otimes 1); (\bullet \otimes g \otimes \rightarrow); (\rightarrow \otimes h).$$

... as structured monoidal category



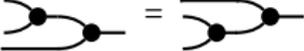
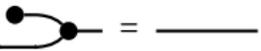
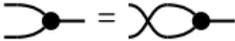
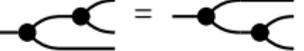
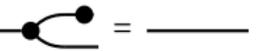
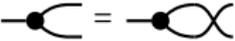
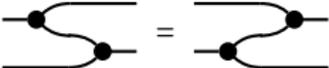
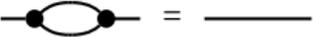
... as structured monoidal category



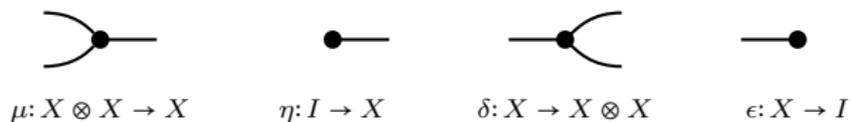
A special commutative Frobenius monoid on  $X$  is

			
$\mu: X \otimes X \rightarrow X$	$\eta: I \rightarrow X$	$\delta: X \rightarrow X \otimes X$	$\epsilon: X \rightarrow I$

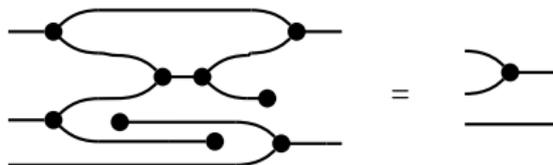
obeying

A special commutative Frobenius monoid on  $X$  is



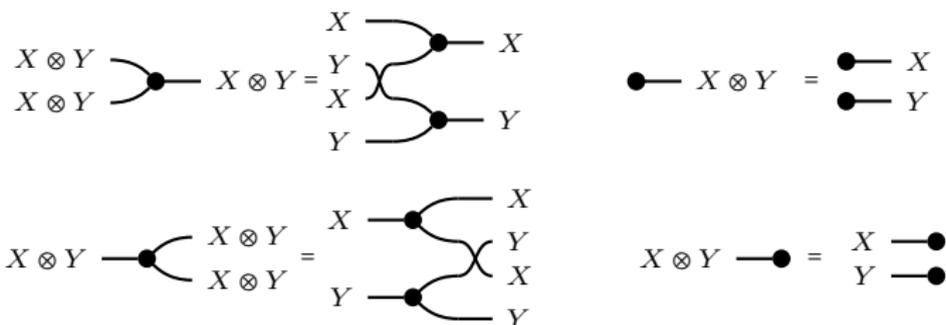
obeying *the spider theorem*



A **hypergraph category** is a symmetric monoidal category in which each object  $X$  is equipped with a Frobenius structure in a way compatible with the monoidal product.

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This means that the Frobenius structure on  $I$  is  $(\rho_I^{-1}, \text{id}_I, \rho_I, \text{id}_I)$  and for all  $X, Y$ , the Frobenius structure on  $X \otimes Y$  is



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The diagram shows four equations defining the Frobenius structure on the tensor product  $X \otimes Y$  in a hypergraph category. Each equation uses a dot to represent the Frobenius multiplication or comultiplication.

- Top-left:** A dot with two inputs labeled  $X \otimes Y$  and one output labeled  $X \otimes Y$  is equal to a dot with two inputs labeled  $X$  and  $Y$ , and two outputs labeled  $X$  and  $Y$ . The inputs  $X$  and  $Y$  are connected to the dot by arcs that cross.
- Top-right:** A dot with one input labeled  $X \otimes Y$  and two outputs labeled  $X$  and  $Y$  is equal to a dot with two inputs labeled  $X$  and  $Y$  and one output labeled  $X \otimes Y$ . The inputs  $X$  and  $Y$  are connected to the dot by arcs that cross.
- Bottom-left:** A dot with one input labeled  $X \otimes Y$  and two outputs labeled  $X$  and  $Y$  is equal to a dot with two inputs labeled  $X$  and  $Y$  and one output labeled  $X \otimes Y$ . The inputs  $X$  and  $Y$  are connected to the dot by arcs that cross.
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A **hypergraph functor** is a strong symmetric monoidal functor  $(F, \varphi)$  such that if  $(\mu_X, \eta_X, \delta_X, \epsilon_X)$  is the Frobenius structure on  $X$ , then  $(\varphi_{X,X}; F\mu_X, \varphi_I; F\eta_X, F\delta_X; \varphi_{X,X}^{-1}, F\epsilon_X; \varphi_I^{-1})$  is the Frobenius structure on  $FX$ .

Let  $\mathbf{Hyp}$  be the 2-category with

**objects:** hypergraph categories

**morphisms:** hypergraph functors

**2-morphisms:** monoidal natural transformations.

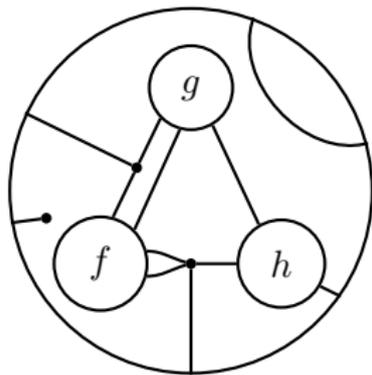
Let  $\mathbf{Hyp}_{\text{OF}}$  be the full sub-2-category of objectwise-free hypergraph categories.

**Theorem** (Coherence for hypergraph categories)

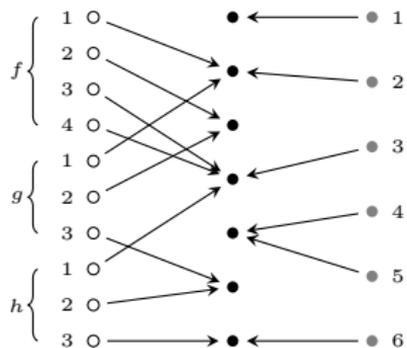
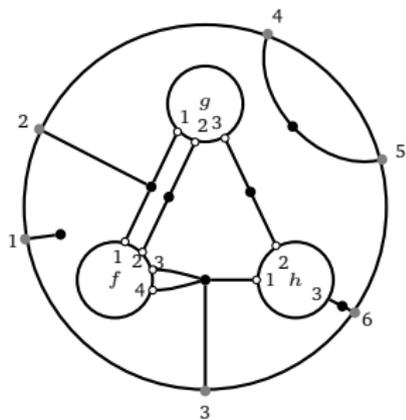
$\mathbf{Hyp}_{\text{OF}}$  and  $\mathbf{Hyp}$  are 2-equivalent.

## II. Cospan algebras

Abstractly, how do we construct this?



... as operad algebra



$$A \longrightarrow N \longleftarrow B$$

Define  $\mathbf{Cospan}_\Lambda = \coprod_{\lambda \in \Lambda} \mathbf{Cospan}(\mathbf{FinSet})$ .

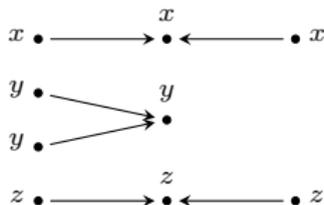
$\mathbf{Cospan}_\Lambda$  is the symmetric monoidal category with

**objects:**  $\Lambda$ -typed finite sets  $t: X \rightarrow \Lambda$ .

**morphisms:** cospans over  $\Lambda$ .

$$\begin{array}{ccccc} X & \xrightarrow{f_1} & N & \xleftarrow{f_2} & Y \\ & \searrow t & \downarrow s & \swarrow u & \\ & & \Lambda & & \end{array}$$

**monoidal product:** disjoint union  $\oplus$



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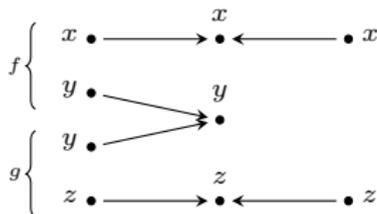
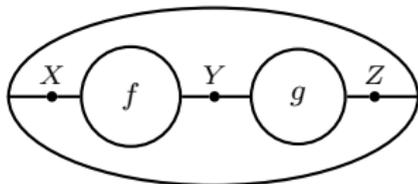
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 X & \xrightarrow{f_1} & N & \xleftarrow{f_2} & Y \\
 & \searrow t & \downarrow s & \swarrow u & \\
 & & \Lambda & & 
 \end{array}$$

**monoidal product:** disjoint union  $\oplus$



Let **CospanAlg** be the category with

**objects:** lax symmetric monoidal functors

$$\Lambda \quad A: (\mathbf{Cospan}_\Lambda, \oplus) \longrightarrow (\mathbf{Set}, \times)$$

**morphisms:** monoidal natural transformations

$$\begin{array}{ccc} \Lambda & \mathbf{Cospan}_\Lambda & \xrightarrow{A} \\ f \downarrow & \mathbf{Cospan}_f \downarrow & \downarrow \alpha \\ \mathbf{List}(\Lambda') & \mathbf{Cospan}_{\Lambda'} & \xrightarrow{A'} \end{array} \mathbf{Set}$$

# III. The equivalence

**Theorem**

$\mathbf{Hyp}_{\text{OF}}$  and  $\mathbf{CospanAlg}$  are (1-)equivalent.

Proof sketch:

1. Work over  $\Lambda$ .
2. Frobenius monoids define cospan algebra.
3. Cospan algebras define homsets of hypergraph categories.

# 1. Working over $\Lambda$

## Lemma

There is a Grothendieck fibration  $\mathbf{Gens}: \mathbf{Hyp}_{\text{OF}} \rightarrow \mathbf{Set}_{\text{List}}$  sending an objectwise-free hypergraph category to its set of generating objects.

This implies

$$\mathbf{Hyp}_{\text{OF}} \cong \int^{\Lambda \in \mathbf{Set}_{\text{List}}} \mathbf{Hyp}_{\text{OF}}(\Lambda)$$

Note also

$$\mathbf{CospanAlg} = \int^{\Lambda \in \mathbf{Set}_{\text{List}}} \mathbf{Lax}(\mathbf{Cospan}_{\Lambda}, \mathbf{Set})$$

## 2. Frobenius defines cospan algebras

### Lemma

$\mathbf{Cospan}_\Lambda$  is the free hypergraph category over  $\Lambda$  (ie. with objects generated by  $\Lambda$ ). That is, there is an adjunction

$$\mathbf{Set}_{\text{List}} \begin{array}{c} \xrightarrow{\text{Cospan}_-} \\ \perp \\ \xleftarrow{\text{Gens}} \end{array} \mathbf{Hyp}_{\text{OF}}$$

Given a hypergraph category  $\mathcal{H}$  over  $\Lambda$ , we can construct a cospan algebra

$$A_{\mathcal{H}}: \mathbf{Cospan}_\Lambda \xrightarrow{\text{Frob}} \mathcal{H} \xrightarrow{\mathcal{H}(I,-)} \mathbf{Set}.$$

### 3. Cospans define hypergraph structure

**Lemma**

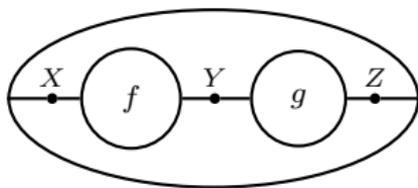
Hypergraph categories are self dual compact closed.

Given a cospan algebra  $A$  over  $\Lambda$ , we may define a hypergraph category  $\mathcal{H}_A$  over  $\Lambda$  with homsets

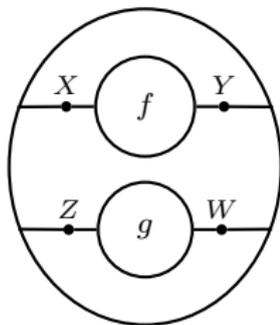
$$\mathcal{H}_A(X, Y) = A(X \oplus Y).$$

### 3. Cospans define hypergraph structure

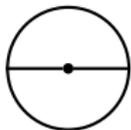
The remaining structure is defined by certain cospans.



composition



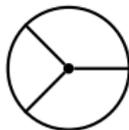
monoidal product



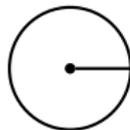
identity



braiding



(co)multiplication



(co)unit

**Theorem** (Coherence for hypergraph categories)  
 $\mathbf{Hyp}_{\text{OF}}$  and  $\mathbf{Hyp}$  are 2-equivalent.

**Theorem**  
 $\mathbf{Hyp}_{\text{OF}}$  and  $\mathbf{CospanAlg}$  are (1-)equivalent.