Problem Set 1

IAP 2019 18.S097: Applied Category Theory

Due Tuesday January 22

Good academic practice is expected. In particular, cooperation is encouraged, but assignments must be written up alone, and collaborators and resources consulted must be acknowledged. Please let us know if you consult the Solutions section in the book.

We suggest that you attempt all problems, but we do not expect all problems to be solved.

Question 1. *Some familiar friends.*

The usual order on the set of natural numbers $\mathbb{N}$ says that $m \leq n$ if there is a natural number d such that $m + d = n$. Another order is the *division order*: we write $n|m$ if n divides perfectly into m. The meet of any two numbers in this poset has a common name, that you may have learned when you were around 10 years old; what is it? Similarly the join of any two numbers has a common name; what is it?

Question 2. *There exists and for all.*

Choose sets X and Y each with at least two elements, and choose a function $f\colon X \to Y$. Recall that we write $\mathsf{P}(A)$ for the powerset of some set A. The function f induces a monotone map $f^*\colon \mathsf{P}(Y) \to \mathsf{P}(X)$ that sends a subset $B \subseteq Y$ to its preimage $f^{-1}(B) \subseteq X$. In the other direction, we may define monotone maps $f_!\colon \mathsf{P}(X) \to \mathsf{P}(Y)$ by

$$f_!(A) \coloneqq \{b \in B \mid \text{there exists } a \in A \text{ such that } f(a) = b\}$$

and

$$f_*(A) \coloneqq \{b \in B \mid \text{for all } a \text{ such that } f(a) = b, \text{ we have } a \in A\}$$

where $A \subseteq X$.

(a) Choose two different subsets $B_1, B_2 \subseteq Y$ and find $f^*(B_1)$ and $f^*(B_2)$.
(b) Choose two different subsets $A_1, A_2 \subseteq X$ and find $f_!(A_1)$ and $f_!(A_2)$.
(c) With the same $A_1, A_2 \subseteq X$, find $f_*(A_1)$ and $f_*(A_2)$.

If you like, prove that $f_!$ is left adjoint to f^* and f_* is right adjoint to f^*.

Question 3. *Picturing(?) Galois connections between total orders.*

Let P and Q be total orders, and $f\colon P \to Q$ and $g\colon Q \to P$ be monotone maps between them. As precisely as you can, describe when f is left adjoint to g. If you're so inclined, you may build upon Exercise 1.99 and Remark 1.100 in the book, but you don't have to.

Question 4. *Gricean Pragmatics (with thanks to Reuben Cohn-Gordon).*

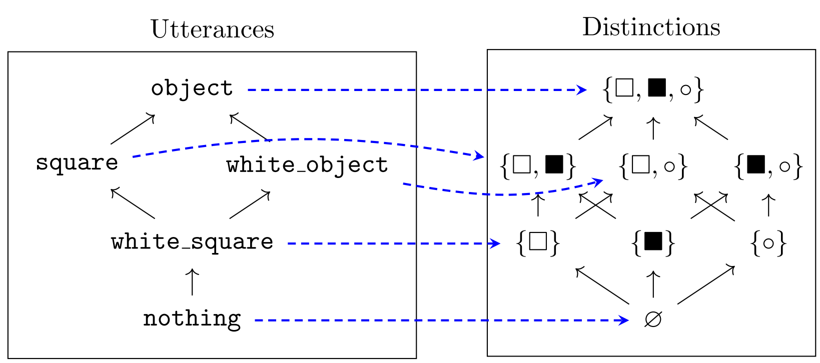
Grice's maxims for cooperative conversation say that a speaker should strive to say the maximally informative utterance that is nonetheless true. A speaker is *pragmatic* if they obey these maxims. We'll model a pragmatic speaker with Galois connections.

Let W be a set of objects. For example, let

$$W = \{\square, \blacksquare, \circ\}.$$

Suppose we know something about an object in the world. This is represented by a subset $W' \subseteq W$. The subset $\{\square, \blacksquare, \circ\}$ represents the knowledge that the object exists, the subset $\{\blacksquare, \circ\}$ represents the knowledge that the object is either $\blacksquare$ or $\circ$, the subset $\varnothing$ represents the knowledge that the object does not exist.

Suppose we also have the poset U of utterances as on the left below. The literal listener L takes an utterance, and understands it as communicating something—a predicate—about an object in question. Let $L\colon U \to \mathsf{P}(W)$ be the monotone map shown in blue:



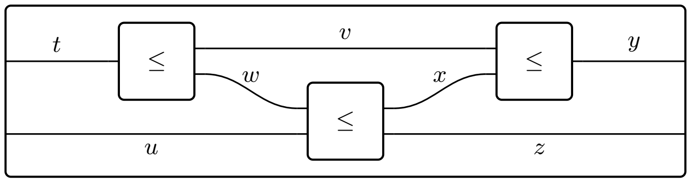
(a) The map L has a left adjoint $S\colon \mathsf{P}(W) \to U$. For each of the eight distinctions $W' \subseteq W$, calculate $S(W)$. For example, what is $S(\{\square\})$, what is $S(\{\blacksquare\})$, etc.?

(b) Explain how S can be understood as the pragmatic speaker.

Extra credit: In fact S happens to have a further left adjoint, which we might call the *pragmatic listener* $L'\colon U \to \mathsf{P}(W)$. Calculate it for extra credit, and explain its semantics in terms of a game where the speaker has a card, either $\square$, $\blacksquare$, or $\circ$, and the pragmatic listener has to guess it.

Question 5. *Diagrammatic proofs.*

Consider the following wiring diagram.



This represents a proof that the inequalities

$$t \leq v + w \qquad w + u \leq x + z \qquad v + x \leq y$$

imply

$$t + u \leq y + z.$$

(a) Formally prove, using only the rules of symmetric monoidal preorders (Definition 2.2 in the textbook), that $t \leq v + w$, $w + u \leq x + z$, and $v + x \leq y$ imply $t + u \leq y + z$.
(b) Reflexivity and transitivity should show up in your proof. Make sure you are explicit about where they do.
(c) Explain how your proof relates to the above wiring diagram.
(d) How can you look at the wiring diagram above and know that the symmetry axiom (Definition 2.2(d)) does not need to be invoked?

Question 6. *The poset of ways.*

Let M be a set and let $\mathcal{M} := (\mathsf{P}(M), \subseteq, M, \cap)$ be the monoidal preorder whose elements are subsets of M. A person gives the following interpretation, "for any set M, imagine it as the set of modes of transportation (e.g. car, boat, foot). Then an $\mathcal{M}$-category $\mathcal{X}$ tells you all the modes that will get you from a all the way to b, for any two points $a, b \in \mathrm{Ob}(\mathcal{X})$."

(a) Draw a graph with three vertices and four or five edges, each labeled with a subset of $M = \{\text{car}, \text{boat}, \text{train}, \text{foot}\}$.
(b) From this graph it is possible to construct an $\mathcal{M}$-category, where the hom-object from x to y is computed as follows: for each path p from x to y, take the intersection of the sets labelling the edges in p. Then, take the union of the these sets over all paths p from x to y. Write out the corresponding three-by-three matrix of hom-objects, and convince yourself that this is indeed an $\mathcal{M}$-category.
(c) Does the person's interpretation look right, or is it subtly mistaken somehow?

Question 7. *Resource theories.*

Tell us about an interesting symmetric monoidal preorder in your own discipline or that you can imagine.

Question 8. *Grade the p-set.*

Give a grade to this problem set, taking into account how much you learned, how interesting or fun it was, and how much time you spent on it. Explain your grade.